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SOME PROPERTIES OF THE EMPIRICAL DISTRIBUTION FUNCTION IN THE
NON-I.I.D. CASE (PART II)

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SOME PROPERTIES OF THE EMPIRICAL DISTRIBUTION FUNCTION IN THE
NON-I.I.D. CASE^{*)} (part II)

Abbreviated title:

ON EMPIRICAL DISTRIBUTION FUNCTIONS

by

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ABSTRACT

Two theorems are proved on empirical df's in the non-i.i.d. case, where moreover the underlying df's need not to be continuous.

These theorems are useful for proving asymptotic normality of rank statistics in the case where the multivariate sample elements are allowed to have different df's and where the scores generating functions are allowed to have a finite number of discontinuities of the first kind. The theorems may also be of interest in their own right.

KEY WORDS & PHRASES: *Empirical distribution, empirical process.*

^{*)} This paper is not for review; it is meant for publication in a journal.

1. INTRODUCTION

Let k be a fixed positive integer and for each $N = 1, 2, \dots$, let $X_{nN} = (X_{1nN}, X_{2nN}, \dots, X_{knN})$, $n = 1, 2, \dots, N$, be N mutually independent k -dimensional random vectors with joint distribution function (df) F_{nN} and marginal df's $F_{1nN}, F_{2nN}, \dots, F_{knN}$. For each N , moreover, let F_N be the joint empirical df based on the N random vectors $X_{1N}, X_{2N}, \dots, X_{NN}$. All random vectors are supposed to be defined on a single probability space $(\Omega, \mathcal{A}, P)$.

We shall also use the notation $\bar{F}_N = N^{-1} \sum_{n=1}^N F_{nN}$ and $\bar{F}_{iN} = N^{-1} \sum_{n=1}^N F_{inN}$ for $i = 1, 2, \dots, k$. Let us observe that $\bar{F}_N$ has all the properties of a k -variate df and that its marginal df's are the $\bar{F}_{iN}$. Finally we define, for $i = 1, 2, \dots, k$, the inverse $\bar{F}_{iN}^{-1}$ by $\bar{F}_{iN}^{-1}(s) = \inf \{x \mid \bar{F}_{iN}(x) \geq s\}$, for $0 \leq s \leq 1$. Note that in the case where $\bar{F}_N$ is continuous we have $\bar{F}_{iN}(\bar{F}_{iN}^{-1}(s)) = s$ for $s \in [0, 1]$.

This paper contains two theorems which are useful for proving asymptotic normality of rank statistics in the case where the multivariate sample elements are allowed to have different df's and where the scores generating functions are allowed to have a finite number of discontinuities of the first kind. The theorems may also be of interest in their own right.

The first theorem is a generalisation to the non-i.i.d. case of a slightly weaker version of a theorem, due to VAN ZWET (lemma 4.4 in [4]). See also [1]. In fact, VAN ZWET proved that in the i.i.d. case theorem 1 holds true, without the factor $(\log(N+1))^{1/2}$ in (1.1). However, we remark that our theorem 1 still is much stronger than, for instance, a generalisation of the Glivenko-Cantelli theorem to the multivariate non-i.i.d. case would be. Theorem 1 makes it possible to handle problems, connected with discontinuities in the scores generating functions of rank statistics.

For any Borel set B in $\mathbb{R}^k$ we shall write $\int_B d\bar{F}_N = \bar{F}_N(B)$ and $\int_B dF_N = F_N(B)$. By an interval in $\mathbb{R}^k$ the product set of k half open (closed on the right) intervals on the line will be meant.

THEOREM 1. Let $I_1, I_2, \dots$ be a sequence of intervals in $\mathbb{R}^k$ and let $I_N^* = \{I_N^*: I_N^* \text{ is an interval contained in } I_N\}$, $N = 1, 2, \dots$. For every $\varepsilon > 0$ and every positive integer k , there exists $M = M(\varepsilon, k)$, such that for every array of k -variate df's $F_{1N}, F_{2N}, \dots, F_{NN}$, $N = 1, 2, \dots$, every sequence $I_1, I_2, \dots$ and every $N = 1, 2, \dots$,

$$(1.1) \quad P \left(\left\{ \sup_{I_N^* \in I_N} |F_N \{I_N^*\} - \bar{F}_N \{I_N^*\}| > M \left(\frac{\log(N+1) \bar{F}_N \{I_N\}^{\frac{1}{2}}}{N} \right) \right\} \right) \leq \varepsilon.$$

The second theorem is a result for the one-dimensional non-i.i.d. case, already given by SEN [5] for continuous underlying df's. However, it is clear from SHORACK [6] that the proof given by SEN is incorrect. An entirely different proof is provided here.

THEOREM 2. For $k = 1$ and every $\varepsilon > 0$ and $\delta \in (0, \frac{1}{2}]$, there exists $M = M(\varepsilon, \delta)$, such that for every array of univariate df's $F_{1N}, F_{2N}, \dots, F_{NN}$, $N = 1, 2, \dots$, and every $N = 1, 2, \dots$,

$$P \left(\left\{ \sup_{x \in \mathbb{R}} \frac{N^{\frac{1}{2}} |F_N(x) - \bar{F}_N(x)|}{(\bar{F}_N(x)(1-\bar{F}_N(x)))^{\frac{1}{2}-\delta}} \geq M \right\} \right) \leq \varepsilon.$$

Our basic tools are two related results of Hoeffding [3]. Suppose that Z_n , $1 \leq n \leq N$, are independent random variables with $P(Z_n=1) = 1 - P(Z_n=0) = p_n$, and suppose that

$$0 < N^{-1} \sum_{n=1}^N p_n = \bar{p} < 1.$$

LEMMA 1 (Hoeffding). If f is a strictly convex function defined on $[0, \infty)$ then

$$E \left(f \left(\sum_{n=1}^N Z_n \right) \right) \leq \sum_{k=0}^N f(k) \binom{N}{k} \bar{p}^k (1-\bar{p})^{N-k},$$

where equality holds if and only if $p_1 = p_2 = \dots = p_N = \bar{p}$.

LEMMA 2 (Hoeffding). *Let b and c be two integers such that*

$$0 \leq b \leq N\bar{p} \leq c \leq N.$$

Then

$$\sum_{n=b}^c \binom{N}{n} \bar{p}^n (1-\bar{p})^{N-n} \leq P \left(b \leq \sum_{n=1}^N Z_n \leq c \right) \leq 1.$$

Both bounds are attained. The lower bound is attained only if $p_1 = p_2 = \dots = p_N = \bar{p}$ unless $b = 0$ and $c = N$.

In section 2 some preliminary results, supplying upper bounds for $\sup |\mathbb{F}_N - \bar{\mathbb{F}}_N|$ and for central moments of $\sum_{n=1}^N Z_n$, are proved. These results are then used in section 3 to prove the theorems in the case of continuous underlying df's. The fact that the theorems also hold in the case of possibly discontinuous df's is immediate from lemma 9, given at the end of section 3.

2. SOME LEMMAS

By $[a]$ we denote the greatest integer in the number a .

LEMMA 3. *Let for $N = 1, 2, \dots$ the k -dimensional df's $F_{1N}, F_{2N}, \dots, F_{NN}$ be continuous and let $I_1, I_2, \dots$ be a sequence of intervals in $\mathbb{R}^k$ with $\bar{\mathbb{F}}_N\{I_N\} > 0$, for $N = 1, 2, \dots$. Define $\bar{\mathbb{F}}_{iN}^{-1}(1+a) = \infty$, for $a > 0$, $i = 1, 2, \dots, k$ and let*

$$I_N^* = \{I_N^*: I_N^* \text{ is an interval contained in } I_N\}$$

and

$$(2.1) \quad \tilde{I}_N = \left\{ \tilde{I}_N: \tilde{I}_N = I_N \cap \bigcap_{i=1}^k \left(\bar{\mathbb{F}}_{iN}^{-1} \left(\frac{n_{i1}}{N} \bar{\mathbb{F}}_N\{I_N\} \right), \bar{\mathbb{F}}_{iN}^{-1} \left(\frac{n_{i2}}{N} \bar{\mathbb{F}}_N\{I_N\} \right) \right) \right\},$$

for k pairs of integers (n_{i1}, n_{i2}) , with $n_{i1} < n_{i2}$ and

$$n_{ij} \in \left\{ 0, 1, 2, \dots, \left[\frac{N}{\bar{\mathbb{F}}_N\{I_N\}} \right] + 1 \right\}, \text{ for } i = 1, 2, \dots, k, j = 1, 2 \}.$$

Then, for each $\omega \in \Omega$, $N = 1, 2, \dots$, $k = 1, 2, \dots$ we have

$$(2.2) \quad \sup_{I_N^* \in I_N^*} |\mathbb{F}_N\{I_N^*\} - \bar{\mathbb{F}}_N\{I_N^*\}| \leq \max_{\tilde{I}_N \in \tilde{I}_N} |\mathbb{F}_N\{\tilde{I}_N\} - \bar{\mathbb{F}}_N\{\tilde{I}_N\}| + 2kN^{-1}\bar{\mathbb{F}}_N\{I_N\}.$$

PROOF. Let I_N^* be an arbitrary interval in I_N . Define

$$\bar{I}_N^* = \bigcap_{\substack{\tilde{I}_N \in \tilde{I}_N \\ \tilde{I}_N \supset I_N^*}} \tilde{I}_N, \quad \underline{I}_N^* = \bigcup_{\substack{\tilde{I}_N \in \tilde{I}_N \\ \tilde{I}_N \subset I_N^*}} \tilde{I}_N.$$

Note that $\bar{I}_N^*$ and $\underline{I}_N^*$ are elements of $\tilde{I}_N \cup \emptyset$ and that

$$(2.3) \quad \bar{F}_N\{\bar{I}_N^*\} - \bar{F}_N\{\underline{I}_N^*\} \leq 2kN^{-1}\bar{F}_N\{I_N\}.$$

If I_N^* is such that $F_N\{I_N^*\} - \bar{F}_N\{I_N^*\} \geq 0$, we have using (2.3)

$$\begin{aligned} |F_N\{I_N^*\} - \bar{F}_N\{I_N^*\}| &= F_N\{I_N^*\} - \bar{F}_N\{I_N^*\} \leq F_N\{\bar{I}_N^*\} - \bar{F}_N\{\underline{I}_N^*\} \leq \\ &\leq F_N\{\bar{I}_N^*\} - \bar{F}_N\{\bar{I}_N^*\} + 2kN^{-1}\bar{F}_N\{I_N\} \leq \\ &\leq |F_N\{\bar{I}_N^*\} - \bar{F}_N\{\bar{I}_N^*\}| + 2kN^{-1}\bar{F}_N\{I_N\} \end{aligned}$$

and if $F_N\{I_N^*\} - \bar{F}_N\{I_N^*\} < 0$, we have

$$\begin{aligned} |F_N\{I_N^*\} - \bar{F}_N\{I_N^*\}| &= \bar{F}_N\{I_N^*\} - F_N\{I_N^*\} \leq \bar{F}_N\{\bar{I}_N^*\} - F_N\{\underline{I}_N^*\} \leq \\ &\leq \bar{F}_N\{\bar{I}_N^*\} - F_N\{\underline{I}_N^*\} + 2kN^{-1}\bar{F}_N\{I_N\} \leq \\ &\leq |\bar{F}_N\{\bar{I}_N^*\} - F_N\{\underline{I}_N^*\}| + 2kN^{-1}\bar{F}_N\{I_N\}. \quad \square \end{aligned}$$

The following lemma gives upper bounds for the central moments of $\sum_{n=1}^N Z_n$, where Z_n , $1 \leq n \leq N$, are independent Bernoulli (p_n) random variables.

LEMMA 4. For every $\alpha > \frac{1}{2}$, there exists $M_\alpha \in (0, \infty)$, such that for $N = 1, 2, \dots$,

$$(2.4) \quad E \left| \sum_{n=1}^N Z_n - N\bar{p} \right|^{2\alpha} \leq M_\alpha (N\bar{p})^\alpha, \quad \text{for } \bar{p} = N^{-1} \sum_{n=1}^N p_n \geq N^{-1}$$

and

$$(2.5) \quad E \left| \sum_{n=1}^N Z_n - N\bar{p} \right|^{2\alpha} \leq M_\alpha N\bar{p}, \quad \text{for } \bar{p} = N^{-1} \sum_{n=1}^N p_n \leq N^{-1}.$$

PROOF. Since $\alpha > \frac{1}{2}$, lemma 1 ensures that it is sufficient to prove lemma 4 in the case where $p_1 = p_2 = \dots = p_N = \bar{p}$. First let us prove (2.4). For

$N = 1$, we have $\bar{p} = 1$, so that the left-hand side of (2.4) is zero. Let $F_{N\bar{p}}(y)$ be the distribution function of $|\sum_{n=1}^N Z_n - N\bar{p}|(N\bar{p}(1-\bar{p}))^{-\frac{1}{2}}$, then using an inequality due to S.N. BERNSTEIN [1, p.578], we have for $y > 0$ that

$$1 - F_{N\bar{p}}(y) = P\left(\left|\sum_{n=1}^N Z_n - N\bar{p}\right| > y\sqrt{N\bar{p}(1-\bar{p})}\right) \leq 2 \exp\left(\frac{-y^2}{2 + \frac{2y}{3\sqrt{N\bar{p}(1-\bar{p})}}}\right).$$

Moreover, for $y \geq 1$ and $\bar{p} \geq N^{-1}$, $N = 2, 3, \dots$, we have

$$(N\bar{p}(1-\bar{p}))^{-\frac{1}{2}} \leq (N/(N-1))^{\frac{1}{2}} \leq 2^{\frac{1}{2}},$$

so that then

$$1 - F_{N\bar{p}}(y) \leq 2 \exp\left(\frac{-y^2}{4y}\right) = 2 \exp\left(-\frac{y}{4}\right).$$

Hence, for $\bar{p} \geq N^{-1}$, $N = 2, 3, \dots$,

$$\begin{aligned} E \left| \frac{\sum_{n=1}^N Z_n - N\bar{p}}{\sqrt{N\bar{p}(1-\bar{p})}} \right|^{2\alpha} &= \int_0^\infty y^{2\alpha} dF_{N\bar{p}}(y) = 2\alpha \int_0^\infty y^{2\alpha-1} (1 - F_{N\bar{p}}(y)) dy \leq \\ &\leq 2\alpha \int_0^1 dy + 4\alpha \int_1^\infty y^{2\alpha-1} \exp\left(-\frac{y}{4}\right) dy, \end{aligned}$$

so that (2.4) is proved.

Let us next concentrate on the proof of (2.5). For $k = 1, 2, \dots, N$, we have $P(\sum_{n=1}^N Z_n = k) \leq \frac{(N\bar{p})^k}{k!}$ and $1 + k \leq e^k$, so that for $\alpha > \frac{1}{2}$, $\bar{p} \leq N^{-1}$, $N = 1, 2, \dots$,

$$\begin{aligned} E \left| \sum_{n=1}^N Z_n - N\bar{p} \right|^{2\alpha} &= |0 - N\bar{p}|^{2\alpha} P\left(\sum_{n=1}^N Z_n = 0\right) + \sum_{k=1}^N |k - N\bar{p}|^{2\alpha} P\left(\sum_{n=1}^N Z_n = k\right) \leq \\ &\leq (N\bar{p})^{2\alpha} + \sum_{k=1}^\infty (k - N\bar{p})^{2\alpha} \frac{(N\bar{p})^k}{k!} \leq \\ &\leq N\bar{p} + \sum_{k=0}^\infty (k+1 - N\bar{p})^{2\alpha} \frac{(N\bar{p})^{k+1}}{(k+1)!} \leq \\ &\leq N\bar{p} + N\bar{p} \sum_{k=0}^\infty (k+1)^{2\alpha} \frac{(N\bar{p})^k}{(k+1)!} \leq N\bar{p} + N\bar{p} \sum_{k=0}^\infty \frac{(e^{2\alpha-1})^k}{k!} = \\ &= N\bar{p}(1 + \exp(e^{2\alpha-1})). \quad \square \end{aligned}$$

3. PROOFS OF THE THEOREMS

PROOF OF THEOREM 1 IN THE CASE OF CONTINUOUS UNDERLYING df's.

If $\bar{F}_N\{I_N\} = 0$, the theorem follows immediately. It proves to be convenient to consider the cases $0 < \bar{F}_N\{I_N\} \leq \frac{8 \log(N+1)}{\varepsilon N}$ and $\bar{F}_N\{I_N\} > \frac{8 \log(N+1)}{\varepsilon N}$, for fixed $0 < \varepsilon < 1$, separately.

First suppose that $0 < \bar{F}_N\{I_N\} \leq \frac{8 \log(N+1)}{\varepsilon N}$, and choose $M = M_1(\varepsilon) = (2/\varepsilon)^{3/2}$, so that

$$(3.1) \quad M \left(\frac{\log(N+1) \bar{F}_N\{I_N\}}{N} \right)^{\frac{1}{2}} \geq M \left(\frac{\varepsilon (\bar{F}_N\{I_N\})^2}{8} \right)^{\frac{1}{2}} \geq \frac{\bar{F}_N\{I_N\}}{\varepsilon}.$$

Moreover, since

$$(3.2) \quad \sup_{I_N^* \in \tilde{I}_N} |\mathbb{F}_N\{I_N^*\} - \bar{F}_N\{I_N^*\}| \leq \max(\mathbb{F}_N\{I_N\}, \bar{F}_N\{I_N\}),$$

we have from (3.1), (3.2) and Markov's inequality that the left-hand side of (1.1) is bounded above by

$$P(\{\max(\mathbb{F}_N\{I_N\}, \bar{F}_N\{I_N\}) \geq \bar{F}_N\{I_N\}/\varepsilon\}) = P(\{\mathbb{F}_N\{I_N\} \geq \bar{F}_N\{I_N\}/\varepsilon\}) \leq \varepsilon.$$

Next we suppose that $\bar{F}_N\{I_N\} > \frac{8 \log(N+1)}{\varepsilon N}$, whereas the underlying df's are assumed to be continuous. Application of lemma 3 shows that for $M > 0$ and $N = 1, 2, \dots$, the left-hand side of (1.1) is bounded above by

$$(3.3) \quad P \left(\left\{ \max_{\tilde{I}_N \in \tilde{I}_N} |\mathbb{F}_N\{\tilde{I}_N\} - \bar{F}_N\{\tilde{I}_N\}| > M \left(\frac{\log(N+1) \bar{F}_N\{I_N\}}{N} \right)^{\frac{1}{2}} - 2kN^{-1} \bar{F}_N\{I_N\} \right\} \right) \leq \\ \leq P \left(\left\{ \max_{\tilde{I}_N \in \tilde{I}_N} |\mathbb{F}_N\{\tilde{I}_N\} - \bar{F}_N\{\tilde{I}_N\}| > M \left(\frac{\log(N+1) \bar{F}_N\{I_N\}}{N} \right)^{\frac{1}{2}} - 2k \left(\frac{\bar{F}_N\{I_N\}}{N} \right)^{\frac{1}{2}} \right\} \right),$$

which, for $M > M_2(k) = 4k(\log 2)^{-\frac{1}{2}}$ and $N = 1, 2, \dots$, is bounded above by

$$(3.4) \quad P \left(\left\{ \max_{\tilde{I}_N \in \tilde{I}_N} |\mathbb{F}_N\{\tilde{I}_N\} - \bar{F}_N\{\tilde{I}_N\}| > \frac{1}{2} M \left(\frac{\log(N+1) \bar{F}_N\{I_N\}}{N} \right)^{\frac{1}{2}} \right\} \right) \leq \\ \leq \sum_{\tilde{I}_N \in \tilde{I}_N} P \left(|\mathbb{F}_N\{\tilde{I}_N\} - \bar{F}_N\{\tilde{I}_N\}| > \frac{1}{2} M \left(\frac{\log(N+1) \bar{F}_N\{I_N\}}{N} \right)^{\frac{1}{2}} \right).$$

Since $\frac{1}{2}M(N \log(N+1) \bar{F}_N\{I_N\})^{\frac{1}{2}} \geq 1$, for $M \geq M_3 = (\sqrt{2} \log 2)^{-1}$, lemma 2 is applicable, so that we may assume that $N \mathbb{F}_N\{\tilde{I}_N\}$ in (3.4) is a binomial rv with parameters N and $\bar{F}_N\{\tilde{I}_N\}$.

Applying Bernstein's inequality [1, p.578] and using $\max(\bar{F}_N\{\tilde{I}_N\}, 1 - \bar{F}_N\{\tilde{I}_N\}) \leq 1$, we find for $M > 0$,

$$(3.5) \quad P \left(\left| \mathbb{F}_N\{\tilde{I}_N\} - \bar{F}_N\{\tilde{I}_N\} \right| \geq \frac{1}{2}M \left(\frac{\log(N+1) \bar{F}_N\{I_N\}}{N} \right)^{\frac{1}{2}} \right) \leq \\ \leq 2 \exp \left(- \frac{\frac{1}{4}M^2 N \log(N+1) \bar{F}_N\{I_N\}}{2N \bar{F}_N\{\tilde{I}_N\} + \frac{1}{3} M(N \log(N+1) \bar{F}_N\{I_N\})^{\frac{1}{2}}} \right).$$

Moreover, since $\bar{F}_N\{I_N\} > \frac{8 \log(N+1)}{\varepsilon N} > \frac{8 \log(N+1)}{N}$ and $\frac{\bar{F}_N\{\tilde{I}_N\}}{\bar{F}_N\{I_N\}} \leq 1$, we obtain for (3.5) the following upper bound

$$(3.6) \quad 2 \exp \left(- \frac{\frac{3}{2} M^2 \sqrt{2} \log(N+1)}{12\sqrt{2} + M} \right) \leq 2 \exp(-\frac{3}{4}M \log(N+1)), \\ \text{for } M \geq M_4 = 12\sqrt{2}.$$

Noting that the number of elements in $\tilde{I}_N$ is bounded above by

$$\left(\frac{N}{\bar{F}_N\{I_N\}} + 4 \right)^{2k} \leq \left(\frac{N^2 \varepsilon}{8 \log(N+1)} + 4 \right)^{2k} \leq (5N^2)^{2k},$$

we obtain from (3.3) - (3.6) for $M \geq \max(M_1, M_2, M_3, M_4) = M_5$ the following upper bound for the left-hand side of (1.1),

$$(5N^2)^{2k} 2(N+1)^{-\frac{3}{4}M} \leq 2.5^{2k} (N+1)^{4k - \frac{3}{4}M} \leq 2.5^{2k} \cdot 2^{4k - \frac{3}{4}M},$$

$$\text{for } M > \max(M_5, 5\frac{1}{3}k). \quad \square$$

Before presenting the proof of theorem 2, in the case of continuous underlying df's, we introduce some more notation. Moreover, from now on it will be assumed that $k = 1$. Define, for $n = 1, 2, \dots, N$,

$$Y_{nN} = \bar{F}_N(X_{nN}) \quad \text{and} \quad G_{nN}(t) = F_{nN}(\bar{F}_N^{-1}(t)).$$

Note that, because of the assumptions, Y_{nN} has continuous df G_{nN} on $[0, 1]$ and also

$$N^{-1} \sum_{n=1}^N G_{nN}(t) = t, \quad \text{for } 0 \leq t \leq 1.$$

Let $\mathbb{G}_N$ denote the empirical df of $Y_{1N}, Y_{2N}, \dots, Y_{NN}$; following SHORACK [6] we shall call $\mathbb{G}_N$ the reduced empirical df of $X_{1N}, X_{2N}, \dots, X_{NN}$. We remark that it suffices to prove theorem 2 for the empirical df $\mathbb{G}_N$. Furthermore, we define the reduced empirical process X_N by

$$X_N(t) = N^{\frac{1}{2}}(\mathbb{G}_N(t) - t), \quad \text{for } 0 \leq t \leq 1,$$

and the process S_N by setting $S_N(\frac{i}{N}) = X_N(\frac{i}{N})$, for $i = 0, 1, \dots, N$, and by letting S_N be linear on the intervals $[\frac{i-1}{N}, \frac{i}{N}]$. Finally, we define on $[0, 1]$ the stochastic processes $X_{N\delta}(t)$ and $S_{N\delta}(t)$ by $X_{N\delta}(t) = X_N(t)/q_\delta(t)$ and $S_{N\delta}(t) = S_N(t)/q_\delta(t)$, where for $0 < \delta \leq \frac{1}{2}$,

$$q_\delta(t) = (t(1-t))^{\frac{1}{2}-\delta}, \quad \text{for } 0 < t < 1,$$

and

$$q_\delta(1) = q_\delta(0) = 1.$$

The following lemmas are needed for the proof of theorem 2:

LEMMA 5. For $k = 1$ and every $\alpha > \frac{1}{2}$ there exists $M_\alpha \in (0, \infty)$ such that for every $0 \leq s, t \leq 1$, every $N = 1, 2, \dots$ and every array of continuous univariate df's $F_{1N}, F_{2N}, \dots, F_{NN}$, $N = 1, 2, \dots$,

$$E |S_N(t) - S_N(s)|^{2\alpha} \leq M_\alpha |t - s|^\alpha.$$

PROOF. (The proof follows the pattern of SHORACK [6], where lemma 5 is proved for $\alpha = 2$.)

Let $N \geq 1$, $0 \leq s \leq t \leq 1$, $\alpha > \frac{1}{2}$ and the continuous univariate df's $F_{1N}, F_{2N}, \dots, F_{NN}$ be arbitrary, but fixed. Choose integers i and j so that

$$\frac{i-1}{N} \leq s \leq \frac{i}{N} \quad \text{and} \quad \frac{j-1}{N} \leq t \leq \frac{j}{N}.$$

Let $\Delta_{km} = |S_N(\frac{m}{N}) - S_N(\frac{k}{N})|$ for integers k and m , with $m > k$. Then from lemma 4 we have for some number $M_\alpha \in (0, \infty)$,

$$\begin{aligned}
E_{\Delta_{km}}^{2\alpha} &= E \left| N^{\frac{1}{2}} \left(\left(\mathbb{G}_N \left(\frac{m}{N} \right) - \mathbb{G}_N \left(\frac{k}{N} \right) \right) - \frac{m-k}{N} \right) \right|^{2\alpha} = \\
&= N^{-\alpha} E \left| N \left(\mathbb{G}_N \left(\frac{m}{N} \right) - \mathbb{G}_N \left(\frac{k}{N} \right) \right) - N \left(\frac{m-k}{N} \right) \right|^{2\alpha} \leq M_{\alpha} \left(\frac{m-k}{N} \right)^{\alpha} = \\
&= e_{km}(\text{say}).
\end{aligned}$$

We also let $\delta_{uv} = |S_N(v) - S_N(u)|$.

Case 1. $i < j - 1$. Then

$$\delta_{st} \leq \Delta_{i,j-1} \vee \Delta_{i,j} \vee \Delta_{i-1,j-1} \vee \Delta_{i-1,j},$$

so that

$$\begin{aligned}
E\delta_{st}^{2\alpha} &\leq e_{i,j-1} + e_{i,j} + e_{i-1,j-1} + e_{i-1,j} \leq 4e_{i-1,j} = \\
&= 4M_{\alpha} \left(\frac{j - (i-1)}{N} \right)^{\alpha} \leq 4M_{\alpha} 3^{\alpha} (t-s)^{\alpha}.
\end{aligned}$$

Case 2. $i = j$. Since the change of a linear function on an interval of length $t - s$ equals the slope times $t - s$, we have

$$\delta_{st} \leq N\Delta_{i-1,i}(t-s),$$

so that

$$E\delta_{st}^{2\alpha} \leq N^{2\alpha} (t-s)^{2\alpha} e_{i-1,i} \leq M_{\alpha} N^{2\alpha} (t-s)^{\alpha} (t-s)^{\alpha} N^{-\alpha} \leq M_{\alpha} (t-s)^{\alpha}.$$

Case 3. $i = j - 1$. Then

$$\delta_{st} \leq \delta_{s, \frac{i}{N}} + \delta_{\frac{i}{N}, t} \leq 2(\delta_{s, \frac{i}{N}} \vee \delta_{\frac{i}{N}, t}),$$

so that by Case 2 we have

$$\begin{aligned}
E\delta_{st}^{2\alpha} &\leq 2^{2\alpha} \left(E\delta_{s, \frac{i}{N}}^{2\alpha} + E\delta_{\frac{i}{N}, t}^{2\alpha} \right) \leq 2^{2\alpha} \left(M_{\alpha} \left(\frac{i}{N} - s \right)^{\alpha} + M_{\alpha} \left(t - \frac{i}{N} \right)^{\alpha} \right) \leq \\
&\leq 2^{2\alpha+1} M_{\alpha} \left\{ \left(\frac{i}{N} - s \right) + \left(t - \frac{i}{N} \right) \right\}^{\alpha} = 2^{2\alpha+1} M_{\alpha} (t-s)^{\alpha}. \quad \square
\end{aligned}$$

LEMMA 6. For $k = 1$ and every $\alpha > \frac{1}{2}$ there exists $M_\alpha \in (0, \infty)$ such that for every $0 \leq s, t \leq 1$, every $N = 1, 2, \dots$, every $\delta \in (0, \frac{1}{2}]$ and every array of continuous univariate df's $F_{1N}, F_{2N}, \dots, F_{NN}$, $N = 1, 2, \dots$,

$$(3.7) \quad E\{|S_{N\delta}(t) - S_{N\delta}(s)|^{2\alpha}\} \leq M_\alpha |t - s|^{2\alpha\delta}.$$

PROOF. Because of symmetry in s and t of (3.7) we suppose $t \geq s$, so that we only have to prove the lemma in the three cases $0 \leq s \leq t \leq \frac{1}{2}$,

$0 \leq s \leq \frac{1}{2} \leq t \leq 1$ and $\frac{1}{2} \leq s \leq t \leq 1$.

Since for the second case we have (c_r -inequality)

$$E|S_{N\delta}(t) - S_{N\delta}(s)|^{2\alpha} \leq c_\alpha E|S_{N\delta}(t) - S_{N\delta}(\frac{1}{2})|^{2\alpha} + c_\alpha E|S_{N\delta}(\frac{1}{2}) - S_{N\delta}(s)|^{2\alpha},$$

where c_α is a number, depending on α only, it suffices to give a proof for the first and third case. Moreover, the lemma is trivially true for s is zero.

Let $\alpha > \frac{1}{2}$, $0 \leq s \leq t \leq \frac{1}{2}$, $N \geq 1$, $\delta \in (0, \frac{1}{2}]$ and the continuous univariate df's $F_{1N}, F_{2N}, \dots, F_{NN}$ be arbitrary, but fixed and let M_i , $i = 1, 2, \dots, 6$ be numbers, only depending on α . Because of lemma 5,

$$\begin{aligned} E\left|\frac{S_N(t)}{q_\delta(t)} - \frac{S_N(s)}{q_\delta(s)}\right|^{2\alpha} &\leq M_1 E\left|\frac{S_N(t) - S_N(s)}{q_\delta(t)}\right|^{2\alpha} + M_1 E\left|\left(\frac{1}{q_\delta(t)} - \frac{1}{q_\delta(s)}\right)S_N(s)\right|^{2\alpha} \leq \\ &\leq M_2 (t-s)^{2\alpha\delta} + M_3 \left(\frac{1}{q_\delta(s)} - \frac{1}{q_\delta(t)}\right)^{2\alpha} s^\alpha. \end{aligned}$$

Moreover, for $s \leq \frac{1}{2}t$, we have

$$s^{\frac{1}{2}} \left(\frac{1}{q_\delta(s)} - \frac{1}{q_\delta(t)}\right) \leq M_4 \frac{1}{s^{\frac{1}{2}-\delta}} s^{\frac{1}{2}} = M_4 s^\delta \leq M_4 (t-s)^\delta,$$

whereas for $s > \frac{1}{2}t$, we find

$$\begin{aligned} s^{\frac{1}{2}} \left(\frac{1}{q_\delta(s)} - \frac{1}{q_\delta(t)}\right) &\leq M_5 s^{\frac{1}{2}} \left(\frac{1}{s^{\frac{1}{2}-\delta}} - \frac{1}{t^{\frac{1}{2}-\delta}}\right) = M_5 s^{\frac{1}{2}} \int_s^t \frac{du}{u^{3/2-\delta}} \leq \\ &\leq M_5 \frac{s^{\frac{1}{2}}(t-s)}{s^{3/2-\delta}} \leq M_6 \frac{(t-s)}{t^{1-\delta}} \leq M_6 (t-s)^\delta. \end{aligned}$$

The proof for the case $\frac{1}{2} \leq s \leq t \leq 1$ goes analogously. $\square$

LEMMA 7. For $k = 1$ and every $\varepsilon > 0$ and $\delta \in (0, \frac{1}{2}]$, there exists $M = M(\varepsilon, \delta)$, such that for every array of continuous univariate df's $F_{1N}, F_{2N}, \dots, F_{NN}$, $N = 1, 2, \dots$, and every $N = 1, 2, \dots$,

$$P\left(\left\{\sup_{0 \leq t \leq 1} |S_{N\delta}(t)| \geq M(\varepsilon, \delta)\right\}\right) \leq \varepsilon.$$

PROOF. Choose arbitrary, but fixed $\varepsilon > 0$, $\delta \in (0, \frac{1}{2}]$, $N \in \{1, 2, \dots\}$ and continuous univariate df's $F_{1N}, F_{2N}, \dots, F_{NN}$. Because of lemma 6 we have for every $0 \leq s, t \leq 1$,

$$(3.8) \quad E\{|S_{N\delta}(t) - S_{N\delta}(s)|^{[\delta^{-1}] + 2}\} \leq M_\delta |t - s|^{\delta([\delta^{-1}] + 2)},$$

where $M_\delta \in (0, \infty)$ only depends on δ . Remark that $\delta([\delta^{-1}] + 2) > 1$.

Consider for fixed positive integer m , the random variables

$$\xi_{im} = S_{N\delta}\left(\frac{i}{m}\right) - S_{N\delta}\left(\frac{i-1}{m}\right) \quad i = 1, 2, \dots, m.$$

Obviously, $S_{N\delta}\left(\frac{k}{m}\right) = \xi_{1m} + \dots + \xi_{km}$ ($S_{N\delta}(0) = 0$).

Because of (3.8) we now have for $0 \leq i \leq j \leq m$, $M \in (0, \infty)$ that

$$\begin{aligned} P\left(\left|S_{N\delta}\left(\frac{j}{m}\right) - S_{N\delta}\left(\frac{i}{m}\right)\right| \geq M\right) &\leq M^{-[\delta^{-1}] - 2} E\left\{\left|S_{N\delta}\left(\frac{j}{m}\right) - S_{N\delta}\left(\frac{i}{m}\right)\right|^{[\delta^{-1}] + 2}\right\} \leq \\ &\leq M^{-[\delta^{-1}] - 2} M_\delta \left(\frac{j}{m} - \frac{i}{m}\right)^{\delta([\delta^{-1}] + 2)} = \\ &= M^{-[\delta^{-1}] - 2} \left(\sum_{i < \ell \leq j} u_\ell\right)^{\delta([\delta^{-1}] + 2)}, \end{aligned}$$

where $u_\ell = (M_\delta)^{(\delta([\delta^{-1}] + 2))} \frac{1}{m}$, for $\ell = 1, 2, \dots, m$.

Applying theorem 12.2 of [2, p.94] we obtain for some $K_\delta \in (0, \infty)$, depending on δ only, that

$$\begin{aligned} P\left(\left\{\max_{0 \leq i \leq m} \left|S_{N\delta}\left(\frac{i}{m}\right)\right| \geq M\right\}\right) &\leq K_\delta M^{-[\delta^{-1}] - 2} (u_1 + u_2 + \dots + u_m)^{\delta([\delta^{-1}] + 2)} = \\ &= K_\delta M^{-[\delta^{-1}] - 2} M_\delta. \end{aligned}$$

Since, for each $\omega \in \Omega$, $S_{N\delta}$ is a continuous function on $[0, 1]$, letting $m \rightarrow \infty$ leads to

$$P\left(\left\{\sup_{0 \leq t \leq 1} |S_{N\delta}(t)| \geq M\right\}\right) \leq K_\delta M^{-[\delta^{-1}] - 2} M_\delta,$$

from which the lemma follows. $\square$

LEMMA 8. For $k = 1$ and every $\varepsilon > 0$ and $\delta \in (0, \frac{1}{2}]$, there exists $M = M(\varepsilon, \delta)$, such that for every array of continuous univariate df's $F_{1N}, F_{2N}, \dots, F_{NN}$, $N = 1, 2, \dots$, and every $N = 1, 2, \dots$,

$$P\left(\left\{\sup_{0 \leq t \leq 1} |X_{N\delta}(t) - S_{N\delta}(t)| \geq M(\varepsilon, \delta)\right\}\right) \leq \varepsilon.$$

PROOF. We shall only give the proof of the lemma in the case where we restrict t in the supremum to the interval $[0, \frac{1}{2}]$ and $N = 2, 3, \dots$.

We remark that for $M > 0$ and $N = 2, 3, \dots$ and $\delta \in (0, \frac{1}{2}]$,

$$(3.9) \quad P\left(\left\{\sup_{0 \leq t \leq \frac{1}{2}} |X_{N\delta}(t) - S_{N\delta}(t)| \geq 4M\right\}\right) \leq P\left(\left\{\sup_{N^{-1} \leq t \leq \frac{1}{2}} |X_{N\delta}(t) - S_{N\delta}(t)| \geq M\right\}\right) +$$

$$+ P\left(\left\{\sup_{0 \leq t \leq N^{-1}} |S_{N\delta}(t)| \geq M\right\}\right) +$$

$$+ P\left(\left\{\sup_{0 \leq t \leq N^{-1}} \frac{N^{\frac{1}{2}} G_N(t)}{q_\delta(t)} \geq M\right\}\right) +$$

$$+ P\left(\left\{\sup_{0 \leq t \leq N^{-1}} \frac{N^{\frac{1}{2}} t}{q_\delta(t)} \geq M\right\}\right).$$

Let us now derive upper bounds for these terms separately, assuming throughout continuous univariate df's $F_{1N}, F_{2N}, \dots, F_{NN}$.

From the fact that $|X_N(t) - S_N(t)| \leq |X_N(t) - S_N(\frac{i}{N})| \vee |X_N(t) - S_N(\frac{i-1}{N})|$ for $\frac{i-1}{N} \leq t \leq \frac{i}{N}$, it follows for $M > 0$, $N = 2, 3, \dots$ and $\delta \in (0, \frac{1}{2}]$ that

$$(3.10) \quad P\left(\left\{\sup_{N^{-1} \leq t \leq \frac{1}{2}} |X_{N\delta}(t) - S_{N\delta}(t)| \geq M\right\}\right) \leq$$

$$\leq \sum_{i=2}^{\left[\frac{N+1}{2}\right]} P\left(NG_N\left(\frac{i}{N}\right) - NG_N\left(\frac{i-1}{N}\right) - 1 \geq N^{\frac{1}{2}} M q_\delta\left(\frac{i-1}{N}\right) - 3\right).$$

Moreover, from Chebyshev's inequality, (3.10) and lemma 4 we find that for $\delta \in (0, \frac{1}{2}]$, $\beta = \beta(\delta) = (2-2\delta)/(1-2\delta) > 2$, $M \geq 6\sqrt{2}$ and $N = 2, 3, \dots$,

$$\begin{aligned}
(3.11) \quad & P\left(\left\{\sup_{N^{-1} \leq t \leq \frac{1}{2}} |X_{N\delta}(t) - S_{N\delta}(t)| \geq M\right\}\right) \leq \\
& \leq \sum_{i=2}^{\left[\frac{N+1}{2}\right]} P\left(\left|N\mathbb{G}_N\left(\frac{i}{N}\right) - N\mathbb{G}_N\left(\frac{i-1}{N}\right) - NN^{-1}\right| \geq \frac{1}{2}MN^{\frac{1}{2}}q_{\delta}\left(\frac{i-1}{N}\right)\right) \leq \\
& \leq \sum_{i=2}^{\left[\frac{N+1}{2}\right]} \left(\frac{1}{2}M\right)^{-\beta} q_{\delta}^{-\beta}\left(\frac{i-1}{N}\right) N^{-\frac{1}{2}\beta} E\left\{\left|N\mathbb{G}_N\left(\frac{i}{N}\right) - N\mathbb{G}_N\left(\frac{i-1}{N}\right) - NN^{-1}\right|^{\beta}\right\} \leq \\
& \leq M_{\frac{1}{2}\beta} \left(\frac{1}{2}M\right)^{-\beta} N^{1-\frac{1}{2}\beta} \frac{1}{N} \sum_{i=2}^{\left[\frac{N+1}{2}\right]} q_{\delta}^{-\beta}\left(\frac{i-1}{N}\right) \leq M_{\frac{1}{2}\beta} \left(\frac{1}{2}M\right)^{-\beta} 2^{1-\frac{1}{2}\beta} K_{\delta},
\end{aligned}$$

where $K_{\delta} \in (0, \infty)$ is a number only depending on δ .

Next, since $S_N(t) = NtS_N(N^{-1})$ for $0 \leq t \leq N^{-1}$, we have for $M > 0$, $\delta \in (0, \frac{1}{2}]$ and $N = 2, 3, \dots$, that (Chebyshev's inequality and lemma 4)

$$\begin{aligned}
(3.12) \quad & P\left(\left\{\sup_{0 \leq t \leq \frac{1}{N}} \frac{|S_N(t)|}{q_{\delta}(t)} \geq M\right\}\right) \leq P\left(\frac{|S_N(N^{-1})|}{q_{\delta}(N^{-1})} \geq M\right) = \\
& = P(|N\mathbb{G}_N(N^{-1}) - NN^{-1}| \geq Mq_{\delta}(N^{-1})N^{\frac{1}{2}}) \leq \\
& \leq \frac{M_1}{M^2 N q_{\delta}\left(\frac{1}{N}\right)} \leq K'_{\delta} M^{-2} 2^{-2\delta},
\end{aligned}$$

where K'_{δ} is a number only depending on δ .

Furthermore we have for $M \geq 2^{\frac{1}{2}}$, $N = 2, 3, \dots$ and $\delta \in (0, \frac{1}{2}]$,

$$\begin{aligned}
(3.13) \quad & P\left(\left\{\sup_{0 \leq t \leq N^{-1}} \frac{N^{\frac{1}{2}}t}{q_{\delta}(t)} \geq M\right\}\right) \leq P\left(\left\{\sup_{0 \leq t \leq \frac{1}{N}} \frac{N^{\frac{1}{2}}t}{t^{\frac{1}{2}-\delta}} \geq M2^{\delta-\frac{1}{2}}\right\}\right) = \\
& = P(N^{-\delta} \geq M2^{\delta-\frac{1}{2}}) \leq P(2^{-\delta} \geq M2^{\delta-\frac{1}{2}}) = 0.
\end{aligned}$$

Finally, we have for $0 \leq t \leq N^{-1}$, $M > 0$, $N = 2, 3, \dots$ and $\delta \in (0, \frac{1}{2}]$ that

$$\frac{\mathbb{G}_N(t)}{q_{\delta}(t)} = \frac{\mathbb{G}_N(t)}{t} \frac{t^{\frac{1}{2}+\delta}}{(1-t)^{\frac{1}{2}-\delta}} \leq \frac{\mathbb{G}_N(t)}{t} 2^{\frac{1}{2}-\delta} \left(\frac{1}{N}\right)^{\frac{1}{2}+\delta}$$

and hence

$$(3.14) \quad P\left(\left\{\sup_{0 \leq t \leq \frac{1}{N}} \frac{N^{\frac{1}{2}} \mathbb{G}_N(t)}{q_\delta(t)} \leq M\right\}\right) \geq P\left(\left\{\sup_{0 \leq t \leq 1} \frac{\mathbb{G}_N(t)}{t} \leq \frac{M 2^\delta}{2^{\frac{1}{2}-\delta}}\right\}\right).$$

In [7]^{*)} it is proved that for each $\eta > 0$, there exists a α , $0 < \alpha < 1$, independent of N and of the continuous df's $F_{1N}, F_{2N}, \dots, F_{NN}$, such that

$$(3.15) \quad P\left(\left\{\sup_{0 \leq t \leq 1} \frac{\mathbb{G}_N(t)}{t} \leq \alpha^{-1}\right\}\right) \geq 1 - \eta.$$

Hence, the lemma follows after combining (3.9), (3.11), (3.12), (3.13) and (3.14), together with the remark above. $\square$

PROOF OF THEOREM 2 IN THE CASE OF CONTINUOUS UNDERLYING df's

Remark that it suffices to prove theorem 2 for the empirical df $\mathbb{G}_N$. Then the theorem follows immediately from lemma 7, lemma 8 and the fact that for $M > 0$,

$$\begin{aligned} P\left(\left\{\sup_{0 \leq t \leq 1} |X_{N\delta}(t)| \geq M\right\}\right) &\leq P\left(\left\{\sup_{0 \leq t \leq 1} |S_{N\delta}(t)| \geq \frac{M}{2}\right\}\right) + \\ &+ P\left(\left\{\sup_{0 \leq t \leq 1} |X_{N\delta}(t) - S_{N\delta}(t)| \geq \frac{M}{2}\right\}\right). \quad \square \end{aligned}$$

The following lemma and its proof make it clear that the theorems also hold in the case where the underlying df's are allowed to be discontinuous.

LEMMA 9. *Let k be a fixed positive integer and let $\mathbb{F}_N$ be the empirical df based on N k -variate sample elements $X_n = (X_{1n}, X_{2n}, \dots, X_{kn})$, $n = 1, 2, \dots, N$, where the X_n are distributed independently according to given, possibly discontinuous df's F_n . There exist N k -variate random vectors $Y_n = (Y_{1n}, Y_{2n}, \dots, Y_{kn})$, $n = 1, 2, \dots, N$, where the Y_n are distributed independently according to continuous df's G_n , such that with probability one*

^{*)} The proof of (3.15) can also be found in Report SW 34/75 of the Department of Mathematical Statistics, Mathematisch Centrum, Amsterdam.

$$(3.16) \quad \sup_{-\infty < x_1, x_2, \dots, x_k < \infty} |\mathbb{G}_N(x_1, x_2, \dots, x_k) - \bar{\mathbb{G}}_N(x_1, x_2, \dots, x_k)| \geq \\ \geq \sup_{-\infty < x_1, x_2, \dots, x_k < \infty} |\mathbb{F}_N(x_1, \dots, x_k) - \bar{\mathbb{F}}_N(x_1, x_2, \dots, x_k)|,$$

where $\mathbb{G}_N$ is the empirical df based on the Y_n , $n = 1, 2, \dots, N$ and $\bar{\mathbb{F}}_N = N^{-1} \sum_{n=1}^N \mathbb{F}_n$, $\bar{\mathbb{G}}_N = N^{-1} \sum_{n=1}^N \mathbb{G}_n$.

PROOF. For $i = 1, 2, \dots, k$ let us denote by F_{in} the i^{th} marginal df of F_n , let $\bar{F}_{iN} = N^{-1} \sum_{n=1}^N F_{in}$ and let $\{\xi_v^{(i)}, v = 1, 2, \dots\}$ be the countable set of discontinuity points of $\bar{F}_{iN}$. This set contains the discontinuity points of each F_{in} , $n = 1, 2, \dots, N$. Moreover, let $p_v^{(i)}$ be the jump at $\xi_v^{(i)}$ of $\bar{F}_{iN}$ and let $\{U_v^{(i)}, v = 1, 2, \dots\}$ be a set of uniform $(0, 1)$ distributed random variables, mutually independent and also independent of the random vectors X_n , $n = 1, 2, \dots, N$.

Since $\sum_v p_v^{(i)} \leq 1$ for $i = 1, 2, \dots, k$, we can define for $n = 1, 2, \dots, N$ the random vector $Y_n = (Y_{1n}, Y_{2n}, \dots, Y_{kn})$ as follows ($\mathbb{1}$ denotes the indicator function):

$$(3.17) \quad Y_{in} = X_{in} + \sum_v \mathbb{1}_{X_{in} > \xi_v^{(i)}} p_v^{(i)} + \sum_v \mathbb{1}_{X_{in} = \xi_v^{(i)}} p_v^{(i)} U_v^{(i)},$$

for $i = 1, 2, \dots, k$,

so that X_n is transformed stochastically to Y_n .

Let G_n be the df of Y_n and let $\mathbb{G}_N$ be the empirical df based on $Y_1, Y_2, \dots, Y_N$. It is clear that all the marginal df's of G_n are continuous and hence G_n is continuous.

From definition (3.17) it is immediate that for $n = 1, 2, \dots, N$

$$(3.18) \quad [X_{in} \leq x_i, \text{ for } i = 1, 2, \dots, k] \iff [Y_{in} \leq x_i + \sum_v \mathbb{1}_{x_i \geq \xi_v^{(i)}} p_v^{(i)},$$

for $i = 1, 2, \dots, k$].

Moreover, from (3.18) it is obvious that (with $\bar{\mathbb{G}}_N = N^{-1} \sum_{n=1}^N G_n$)

$$\bar{\mathbb{F}}_N(x_1, x_2, \dots, x_k) = \bar{\mathbb{G}}_N\left(x_1 + \sum_v \mathbb{1}_{x_1 \geq \xi_v^{(1)}} p_v^{(1)}, \dots, x_k + \sum_v \mathbb{1}_{x_k \geq \xi_v^{(k)}} p_v^{(k)}\right)$$

and with probability one

$$\mathbb{F}_N(x_1, x_2, \dots, x_k) = \mathbb{G}_N\left(x_1 + \sum_v \mathbb{1}_{x_1 \geq \xi_v^{(1)}} p_v^{(1)}, \dots, x_k + \sum_v \mathbb{1}_{x_k \geq \xi_v^{(k)}} p_v^{(k)}\right).$$

so that (3.16) holds. $\square$

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